12

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RELIABILITY ACCEPTANCE SAMPLING PLANS BASED UPON PRIOR DISTRIBUTION
Risk Criteria and Their Interpretation

Syracuse University

ROME AIR DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
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This report provides all the necessary information to make practical application of acceptance tests based on a prior distribution, when an exponential distribution of failures and an inverted gamma prior distribution are assumed. Volume I, "Introduction and Problem Definition," presents an introduction to the problem and a summary of the other volumes. Volume II, "Risk Criteria and Their Interpretation," defines the different statistical risks which can be used to characterize the tests and provide a guide for selecting appropriate risks in various practical situations. Volume III, "Implications and

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Determination of the Prior Distribution," provides the means for determining the parameters of the prior distribution from existing data, and discusses the reason for using an inverted gamma. Volume IV, "Design of Testing Plans," provides instructions for establishing a test time and number of allowable failures based on the prior distribution and the selected risks. Volume V, "Sensitivity Analyses," shows the effects on the test parameters caused by changes in the prior parameters.

Block 7. (Continued)

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PREFACE

This report is one of a set of five presenting the results of part of the work done under contract number F-30602-71-C-0312. The report is delivered to RADC in accordance with item A006 of the Contract Data Requirement List. Sponsorship and technical direction of this task originated in the Reliability and Maintainability Engineering Section (A. Coppola, Chief), Reliability Branch (D. Barber, Chief), within the Reliability and Compatibility Division (J. Naresky, Chief) of the Rome Air Development Center. Mr. Anthony Coppola was the Project Engineer who was technically supported by Mr. Jerome Klion.

The titles of the reports on the subject "Reliability Acceptance Sampling Plans Based Upon Prior Distribution" are as follows:

Volume I.	Introduction and Problem Definition.
Volume II.	Risk Criteria and Their Interpretation.
Volume III.	Implications and Determination of the Prior Distribution.
Volume IV.	Design of Testing Plans.
Volume V.	Sensitivity Analyses.

ABSTRACT

This report deals with the definitions, interpretations and appropriateness of producer's and consumer's risks for the design of reliability acceptance sampling plans based upon prior distribution. The general approach presented here is applicable to any failure density and prior distribution. The specific case when the time to failure has an exponential distribution with parameter θ and θ has an inverted gamma prior distribution has been dealt with in detail. The definitions and the mathematical, physical and graphical interpretations of the various risks are discussed and the risks are evaluated from the viewpoint of their meaningfulness to the producer and the consumer. The similarities and differences between acceptance sampling and demonstration, the implications behind the choice of minimum acceptable MTBF and specified MTBF, the basic interests of the producer and the consumer, and the choice of risks that reflect the primary interests of the producer and the consumer for various situations of practical interest are discussed.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Julius Adelsberg, Harold Balaban, Saul Blumenthal and Thomas Drnas for their comments and suggestions for improving this report; and to Anthony Coppola, Anthony Feduccia, and Jerome Klion of RADC for their helpful suggestions. We are particularly indebted to Donald Wu for his valuable programming help and other assistance during the course of this project.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

This report deals with the definitions, interpretations and appropriateness of producer's and consumer's risks for the design of reliability acceptance sampling plans based upon prior distribution. The general approach presented here is applicable to any failure density and prior distribution. For purposes of discussion, however, we consider the specific case when the time to failure, t, has an exponential distribution with parameter θ, i.e.

$$f(t|\theta) = \theta^{-1}. \exp(-t/\theta), \theta > 0, t \geq 0, \tag{1}$$

and θ has an inverted gamma prior distribution with parameters (γ,λ) i.e. $Ga'(\gamma,\lambda)$ given by

$$g(\theta) = \gamma^{\lambda}. \theta^{-(\lambda+1)}. \exp(-\gamma/\theta) /\Gamma\lambda;\ \theta,\gamma,\lambda > 0. \tag{2}$$

The report is divided into two sections. The first section contains the definitions and the mathematical, physical and graphical interpretations of the various risks. The exact protection provided to the producer and the consumer by the use of these risks is discussed. The risks are interrelated i.e. a numerical specification of any pair of producer-consumer risks implies specific calculable values for other such pairs. These interrelationships are analytically and numerically explored. The second section contains an evaluation of the risks from the viewpoint of their meaningfulness to the producer and the consumer. Toward this end, the similarities and differences between acceptance sampling and demonstration, the implications behind the choice of θ_1 (minimum acceptable MTBF) and θ_0 (specified MTBF),

the basic interests of the producer and the consumer, and the choice of a pair of risks that reflects the primary interests of the producer and the consumer for various situations of practical interest are discussed.

The following quantities are of interest and have been considered in detail in this report.

1. The probability $P(R|\theta = \theta_0) = \alpha$ that a system with specified MTBF is rejected.
2. The probability $P(A|\theta = \theta_1) = \beta$ that a system (lot) with minimum acceptable MTBF is accepted.
3. The probability $P(A|\theta \leq \theta_1) = \bar{\beta}$ that a system which is of unacceptable reliability is accepted,
4. The probability $P(R|\theta \geq \theta_0) = \bar{\alpha}$ that a system of acceptable reliability is rejected.
5. The probability $P(\theta \leq \theta_1|A) = \beta^*$ that the MTBF of a system which has been accepted is less than the minimum acceptable MTBF.
6. The probability $P(\theta \leq \theta_0|A) = \beta^{**}$ that the MTBF of a system which has been accepted is less than the specified MTBF.
7. The probability $P(\theta \geq \theta_0|R) = \alpha^*$ that the MTBF of a system which has been rejected is greater than the specified MTBF.
8. The probability $P(R)$ that a randomly selected system is rejected.
9. The probability $P(\theta \leq \theta')$ that a-priori, a randomly selected system has MTBF which does not exceed θ'.

10. The probability $P(\theta \leq \theta' | r)$ that MTBF does not exceed θ' for a system where testing has produced exactly r failures.

11. The mean $E(\theta | A)$ of the MTBF in the accepted systems.

12. The variance $Var(\theta | A)$ of the MTBF in the accepted systems.

2. INTERPRETATION OF RISKS

2.1. Classical Risks (α, β)

The classical producer's risk α and consumer's risk β are defined as follows:

$$\alpha = P(R|\theta = \theta_0) \tag{3}$$

= Probability of rejecting a system whose MTBF is equal to the specified value, θ_0

$$\beta = P(A|\theta = \theta_1) \tag{4}$$

= Probability of accepting a system whose MTBF is equal to the minimum acceptable value.

The (α, β) risks represent two points on the classical operating characteristic (O.C) curve which is a plot of $P(A|\theta)$ versus θ. Such a curve is shown in Fig. 1 for various values of θ (Explanation of other curves in Fig. 1 is given in the sequel). These risks do not provide an explicit control on the probability of acceptance for values of θ other than θ_1 and θ_0. However, $P(A|\theta)$ increases monotonically with θ. Hence, if $\theta > \theta_0$, the probability of rejection is less than α. If $\theta < \theta_1$, the probability of acceptance is less than β. The shape of the O.C. curve governs the degree of protection provided in the indifference zone between θ_1 and θ_0 as can be seen from the $P(A|\theta)$ plot in Figure 1.

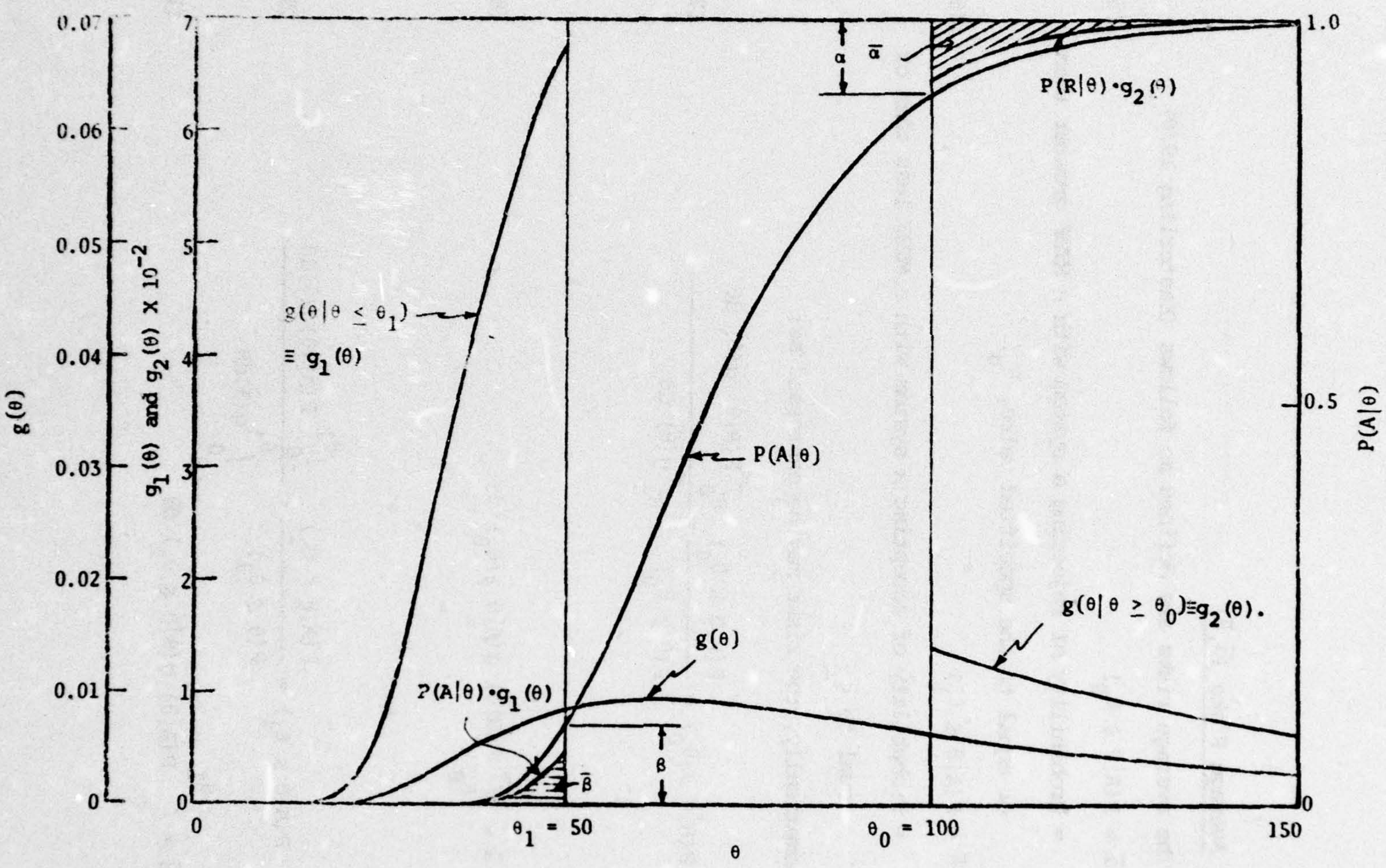

Fig. 1. A GRAPHICAL REPRESENTATION OF VARIOUS RISKS FOR THE PLAN T=950, r*=13, DISCRIMINATION RATIO = 2

2.2 Average Risks ($\bar{\alpha}, \bar{\beta}$)

The average risks are defined as follows (Easterling 1970)

$$\bar{\alpha} = P(R|\theta \geq \theta_0) \tag{5}$$

= Probability of rejecting a system with a MTBF greater than or equal to the specified value, θ_0.

$$\bar{\beta} = P(A|\theta \leq \theta_1) \tag{6}$$

= Probability of accepting a system with a MTBF less than or equal to θ_1

Mathematically, the risks may be expressed as:

$$P(R|\theta \geq \theta_0) = \frac{P(R, \theta \geq \theta_0)}{P(\theta \geq \theta_0)} = \frac{\int_{\theta_0}^{\infty} P(R|\theta)\, g(\theta)\, d\theta}{\int_{\theta_0}^{\infty} g(\theta)\, d\theta} \tag{7}$$

or

$$\bar{\alpha} = \int_{\theta_0}^{\infty} P(R|\theta)\, g(\theta|\theta \geq \theta_0)\, d\theta \tag{8}$$

and

$$P(A|\theta \leq \theta_1) = \frac{P(A, \theta \leq \theta_1)}{P(\theta \leq \theta_1)} = \frac{\int_0^{\theta_1} P(A)|\theta) g(\theta) d\theta}{\int_0^{\theta_1} g(\theta) d\theta} \tag{9}$$

$$\text{or } \bar{\beta} = \int_0^{\theta_1} P(A|\theta)\, g(\theta|\theta \leq \theta_1)\, d\theta \tag{10}$$

Now let us consider the expression for $\bar{\alpha}$ in Equation (8). We know that $P(R|\theta)$ is a monotonically decreasing function of θ and

$$\max_{\theta \in (\theta_0, \infty)} P(R|\theta) = P(R|\theta_0)$$

Therefore

$$\int_{\theta_0}^{\infty} P(R|\theta)\, g(\theta|\theta \geq \theta_0) d\theta \leq \int_{\theta_0}^{\infty} P(R|\theta_0) \cdot g(\theta|\theta \geq \theta_0) d\theta$$

or $$\bar{\alpha} \leq P(R|\theta_0) = \alpha \qquad (11)$$

Similarly, since $P(A|\theta)$ is a monotonically increasing function of θ, we have from Equation (10),

$$\int_0^{\theta_1} P(A|\theta) \cdot g(\theta|\theta \leq \theta_1) d\theta \leq \int_0^{\theta_1} P(A|\theta_1) \cdot g(\theta|\theta < \theta_1) d\theta$$

or $$\bar{\alpha} \leq P(A|\theta_1) = \beta \qquad (12)$$

From Equations (11) and (12), $\bar{\alpha} \leq \alpha$ and $\bar{\beta} \leq \beta$. The equality holds if and only if θ has a Bernoulli distribution with $P(\theta = \theta_0) = \pi$ and $P(\theta = \theta_1) = 1 - \pi$.

A graphical representation of these risks is given in Figure 1. In order to obtain these risks we first get the conditional distributions $g_1(\theta) \equiv g(\theta|\theta \leq \theta_1)$ and $g_2(\theta) \equiv g(\theta|\theta \geq \theta_0)$ from $g(\theta)$. Then the curves $P(R|\theta) \cdot g(\theta_2)$ and $P(A|\theta) \cdot g(\theta_1)$ are obtained from the $P(A|\theta)$ and the $g_2(\theta)$ and $g_1(\theta)$ curves respectively. The risks $\bar{\alpha}$ and $\bar{\beta}$ are the areas under the curves $P(R|\theta) \cdot g_2(\theta)$ and $P(A|\theta) \cdot g_1(\theta)$ respectively, as shown in Figure 1.

The frequency definition of these risks may be given as follows.

$$\bar{\alpha} = \frac{P(R, \theta \geq \theta_0)}{P(\theta \geq \theta_0)} = \frac{\text{Number of good systems that are rejected}}{\text{Total number of good systems}} \tag{13}$$

$$\bar{\beta} = \frac{P(A, \theta \leq \theta_1)}{P(\theta \leq \theta_1)} = \frac{\text{Number of bad systems that are accepted}}{\text{Total number of bad systems}} \tag{14}$$

The meaning of $\bar{\alpha}$, $\bar{\beta}$ can also be appreciated by considering the entries in Table I.

In this table A and R refer to acceptance and rejection decisions and the domain of θ has been divided into three regions, $\theta \geq \theta_0$, $\theta_1 < \theta < \theta_0$ and $\theta \leq \theta_1$. The probabilities $p_1, p_2 \ldots p_6$ are the joint probabilities for the events as indicated in the table, e.g., p_1 is the joint probability of acceptance and of $\theta \geq \theta_0$. Thus, we have

$$\bar{\alpha} = \frac{p_3}{p_1+p_3} \text{ and } \bar{\beta} = \frac{p_2}{p_2+p_4}$$

Note that $P(A)$, $P(\theta \geq \theta_0)$, and $P(\theta \leq \theta_1)$ are not independent events, and hence the numerical values of $p_1, p_2, \ldots p_6$ cannot be obtained by simply knowing $P(A)$, $P(\theta \geq \theta_0)$ and $P(\theta \leq \theta_1)$.

TABLE 1

FREQUENCY ILLUSTRATION OF SOME RISKS

Action \ θ	$\theta \geq \theta_0$	$\theta_1 < \theta < \theta_0$	$\theta \leq \theta_1$	
A	p_1	p_5	p_2	$p_1+p_2+p_5$
R	p_3	p_6	p_4	$p_3+p_4+p_6$
SUMS	$p_1 + p_3$	$p_5 + p_6$	$p_2 + p_4$	$\sum_{i=1}^{6} p_i = 1$

The average risks provide the following protection. If the producer produces a large number of systems, then, in the long run, less than $\bar{\alpha}$ percent of the good systems will be rejected. If the consumer buys a large number of systems, then, in the long run, less than 100 $\bar{\beta}$ percent of the bad systems will be accepted. No explicit control on the probability of acceptance is provided at any specific value of θ when we use the average risk criteria. A graphical representation of these and other risks is shown in Figure 2, and is discussed in Section 2.7.

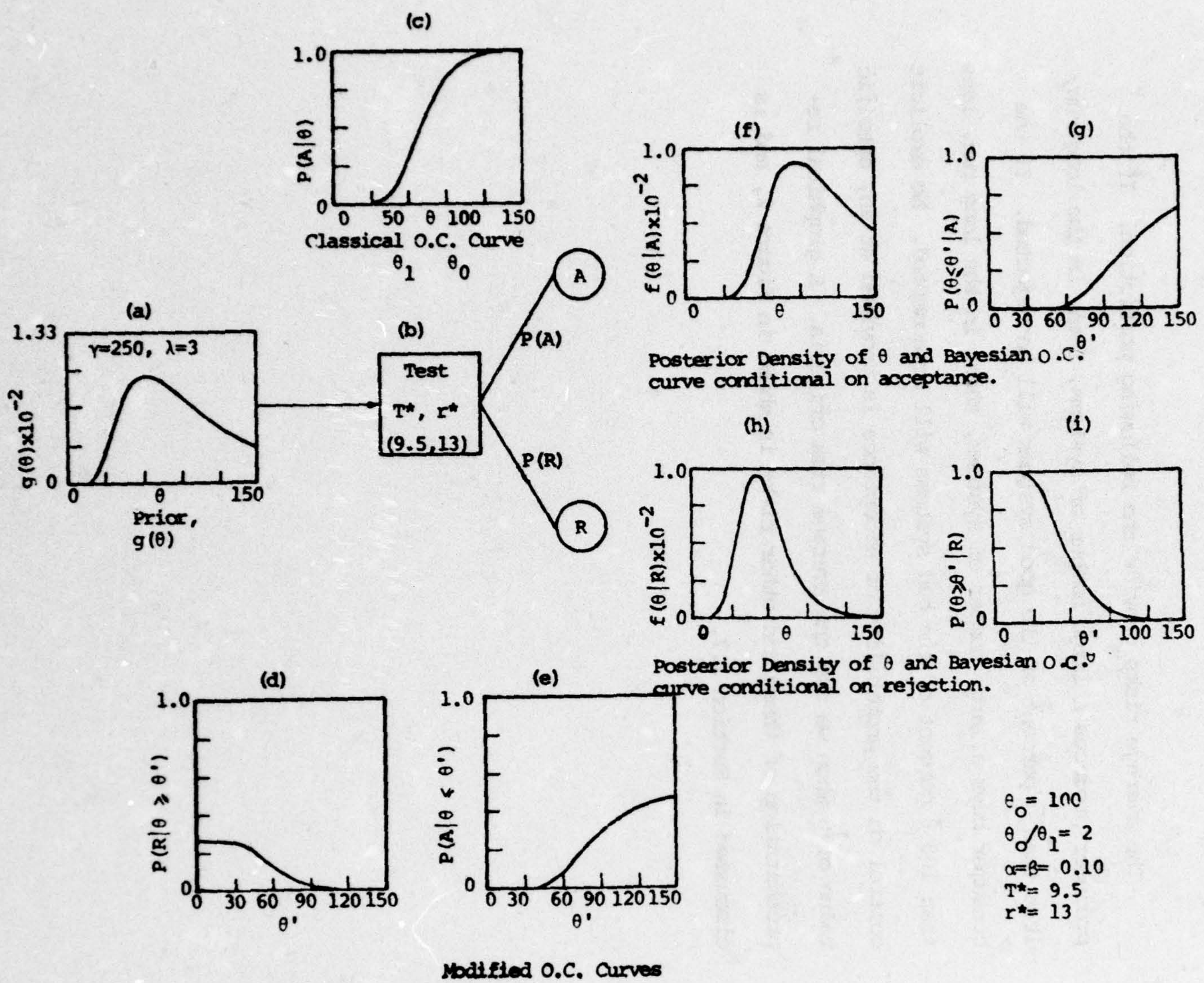

Fig. 2: SEQUENCE OF RISK DEFINITIONS

2.3 Posterior Risks (α^*, β^*)

The (α^*, β^*) risks are defined as follows (Schick and Drnas, 1972)

$$\alpha^* = P(\theta \geq \theta_o | R) \tag{15}$$

This risk is the long run probability of a rejected system being good.

$$\beta^* = P(\theta \leq \theta_1 | A) \tag{16}$$

This risk is the long run probability of an accepted system being bad.

In the frequency sense,

$$\alpha^* = \frac{\text{Number of good systems among the rejected systems}}{\text{Total number of rejected systems}} \tag{17}$$

$$\beta^* = \frac{\text{Number of bad systems among the accepted systems}}{\text{Total number of accepted systems}} \tag{18}$$

Referring to Table 1, we see that

$$\alpha^* = \frac{P(R, \theta \geq \theta_o)}{p(R)} = \frac{p_3}{p_3+p_4+p_6}, \text{ and}$$

$$\beta^* = \frac{P(A, \theta \leq \theta_1)}{p(A)} = \frac{p_2}{p_1+p_2+p_5} .$$

Mathematically,

$$\alpha^* = P(\theta \geq \theta_o | R) = \int_{\theta_0}^{\infty} f(\theta | R) d\theta = \frac{\int_{\theta_o}^{\infty} P(R|\theta) g(\theta) d\theta}{\int_o^{\infty} P(R|\theta) g(\theta) d\theta} \tag{19}$$

$$\beta^* = P(\theta \leq \theta_1 | A) = \int_o^{\theta_1} f(\theta | A) d\theta = \frac{\int_o^{\theta_1} P(A|\theta) g(\theta) d\theta}{\int_o^{\infty} P(A|\theta) g(\theta) d\theta} \tag{20}$$

The (α^*, β^*) risks are shown graphically in Figure 2 and represent a point each on the cumulative distribution functions $P(\theta \geqslant \theta'|R)$ and $P(\theta \leqslant \theta'|A)$ respectively. (Other curves are discussed in Section 2.7)

If the exact number of failures r is known, then the posterior probabilities can be computed as follows

$$P(\theta \geq \theta_o|r) = \frac{P(r|\theta \geq \theta_o) \quad P(\theta \geq \theta_o)}{P(r)} = \frac{P(r,\theta \geq \theta_o)}{P(r)} \tag{21}$$

$$= \frac{\text{Number of good systems among those that experience r failures}}{\text{Number of systems that experience r failures.}}$$

$$P(\theta \leq \theta_1|r) = \frac{P(r|\theta \leq \theta_1) \ P(\theta \leq \theta_1)}{P(r)} = \frac{P(r,\theta \leq \theta_1)}{P(r)} \tag{22}$$

$$= \frac{\text{Number of bad systems among those that experience r failures}}{\text{Number of systems that experience r failures.}}$$

The quantities $P(\theta \geq \theta_o|r)$ and $P(\theta \leqslant \theta_1|r)$ cannot be computed without knowing the test results and hence are not suitable as risk criteria for plan design. They are related to α^* and β^*. For example, if r^* denotes the acceptance number (i.e. $r \leqslant r^*$ implies acceptance), then

$$\beta^* = P(\theta \leq \theta_1 | A) = \frac{P(A | \theta \leq \theta_1) \cdot P(\theta \leq \theta_1)}{P(A)}$$

$$= \sum_{r=o}^{r^*} \frac{P(r | \theta \leq \theta_1) \quad P(\theta \leq \theta_1)}{\sum_{r=0}^{r^*} P(r)}$$

$$= \sum_{r=o}^{r^*} \frac{P(r | \theta \leq \theta_1) \quad P(\theta \leq \theta_1)}{P(r)} \cdot \frac{P(r)}{\sum_{r=0}^{r^*} P(r)}$$

or

$$\beta^* = \sum_{r=o}^{r^*} P(\theta \leq \theta_1 | r) \cdot \frac{P(r)}{\sum_{r=o}^{r^*} P(r)} \tag{23}$$

Therefore β^* represents a weighted sum of $P(\theta \leq \theta_1 | r)$ with weights equal to the probability of occurrence of r in the accepted systems. A similar result holds for α^*.

The following relationships exist between (α^*, β^*) and $(\bar{\alpha}, \bar{\beta})$.

$$P(\theta \geq \theta_o | R) = \frac{P(R | \theta \geq \theta_o) \cdot P(\theta \geq \theta_o)}{P(R)} \tag{24}$$

Hence,

$$\alpha^* = \bar{\alpha} \cdot \frac{P(\theta \geq \theta_o)}{P(R)} \tag{25}$$

Similarly,

$$\beta^* = \bar{\beta} \cdot \frac{P(\theta \leq \theta_1)}{P(A)} \tag{26}$$

If the prior is favorable (if more probability mass is attached to larger values of θ), then $P(\theta \geq \theta_o) > P(R)$ and $P(\theta \leq \theta_1) < P(A)$. Hence, for a favorable prior, $\alpha^* > \bar{\alpha}$ and $\beta^* < \bar{\beta}$. The inequalities will be reversed if the prior is unfavorable to the consumer. The risks provide the following protection. If the producer produces a large number of systems then, in the long run, less than 100 α^* percent of the rejected systems will be good. If the consumer buys a large number of systems then, in the long run, less than 100 β^* percent of the accepted systems will be bad. No explicit control is provided on the probability of acceptance for any specific value of θ.

2.4 Probability of Rejection P(R)

This is a single number given by (Schafer 1973)

$$P(R) = \int_0^\infty P(R|\theta)\ g(\theta)\ d\theta \tag{27}$$

or

$$P(R) = 1 - P(A) = 1 - \int_0^\infty P(A|\theta)\ g(\theta)\ d\theta \tag{28}$$

Note that the integration is over the entire range of θ and specification of θ_o, which is usually specified in conjunction with a producer's risk, is unnecessary.

In the frequency sense we have

$$P(R) = \frac{\text{Total number of systems rejected}}{\text{Total number of systems tested}}$$

For the producer this criterion implies that, in the long run, less than $(100)\cdot P(R)$ percent of the systems will be rejected.

2.5 Alternate Posterior consumer's risk β^{**}

We define a new risk associated with the posterior distribution of θ as follows

$$\beta^{**} = P(\theta \leqslant \theta_o | A) = \int_o^{\theta_0} f(\theta|A)\, d\theta \tag{29}$$

or

$$\beta^{**} = \frac{\int_o^{\theta_0} P(A|\theta)\, g(\theta)\, d\theta}{\int_o^{\infty} P(A|\theta)\, g(\theta)\, d\theta} \tag{30}$$

This risk gives the long run probability of the accepted product having a θ below the specified MTBF θ_o.

2.6 Moments of $f(\theta|A)$

$f(\theta|A)$ represents the distribution of θ in the accepted lots and its moments are of interest to the consumer in determining the MTBF in such lots.

The n^{th} moment of $f(\theta|A)$ is given by

$$E(\theta^n|A) = \int_\theta \theta^n f(\theta|A) d\theta \tag{31}$$

For exponential failure distribution and inverted gamma prior given by

$$g(\theta) = \frac{\gamma^{\lambda}}{\Gamma\lambda} \cdot \theta^{-(\lambda+1)} e^{\gamma/\theta}, \quad \gamma, \lambda, \theta > 0 \tag{32}$$

moments up to $n = [\lambda]$ exist and are given by

$$E(\theta^n|A) = \sum_{r=0}^{r^*} \frac{\Gamma(r+\lambda-n)}{\Gamma(\lambda)\ \Gamma(r+1)} \ \frac{\left(\frac{T}{T+\gamma}\right)^r \left(\frac{\gamma}{T+\gamma}\right)^{\lambda} (T+\gamma)^n}{P(A)} \tag{33}$$

where T and r* are the test time and acceptance number respectively. From Equation (33), the mean and variance of $(\theta|A)$ can be easily computed.

2.7 Sequence of Risk Definitions

Figure 2 on page 10 is a graphical representation of various operating characteristic and other curves as they arise in going from the submitted systems to the accepted and rejected systems. The plot of $g(\theta)$ in (a) represents the prior distribution of θ, an inverted gamma with $\gamma = 250$, and $\lambda = 3$. Systems with this $g(\theta)$ are submitted to a testing plan T = 9500 hours or $T^* = T/\theta_0 = 9.5$ and $r^* = 13$ in (b). For illustration purposes, the plan has been designed for a discrimination ratio $\theta_0/\theta_1 = 2$, and risks $\alpha = \beta = 0.10$. The classical O.C. curve in (c) represents the probability of acceptance for various values of θ. The risks $\alpha = 0.10$ and $\beta = 0.10$ correspond to the values $\theta_0 = 100$ and $\theta_1 = 50$ respectively. The modified O.C. curves in (d) and (e) are the plots of $P(R|\theta \geq \theta')$ and $P(A|\theta \leq \theta')$, respectively, for various values of θ'. For the example being considered, the ordinate in (d) at $\theta' = \theta_0 = 100$ is $\bar{\alpha}$. Similarly, the ordinate in (e) at $\theta' = \theta_1 = 50$ is $\bar{\beta}$. Plots in (f) and (g) are the posterior density and the cumulative posterior distribution of θ conditional on acceptance. Similarly, the plots in (h) and (i) represent the posterior density and the cumulative posterior distribution of θ conditional on rejection. The Bayesian risk β^* is the ordinate in (g) at $\theta' = \theta_1 = 50$. The risk α^* is the ordinate in (i) at $\theta' = \theta_0 = 100$.

3. NUMERICAL EXAMPLES

Example 1:

The different risks are first computed for the case where θ has a Bernoulli distribution, i.e. for the case of a two point prior given by

$$P(\theta=\theta_0)= \pi$$

$$P(\theta=\theta_1)= 1 - \pi$$

The sequence in which the risks are defined is shown in Fig. 3. If α and β are specified, then it is easy to see that

$$\bar{\alpha}= \alpha$$

$$\bar{\beta}= \beta$$

$$P(A)= \pi(1-\alpha) + (1-\pi)\beta$$

$$P(R)= \pi\alpha + (1-\pi)\ (1-\beta)$$

and

$$\alpha^*= \frac{\alpha\pi}{\pi\alpha + (1-\pi)\ (1-\beta)}$$

and

$$\beta^*= \frac{\beta(1-\pi)}{\pi(1-\alpha) + (1-\pi)\beta}$$

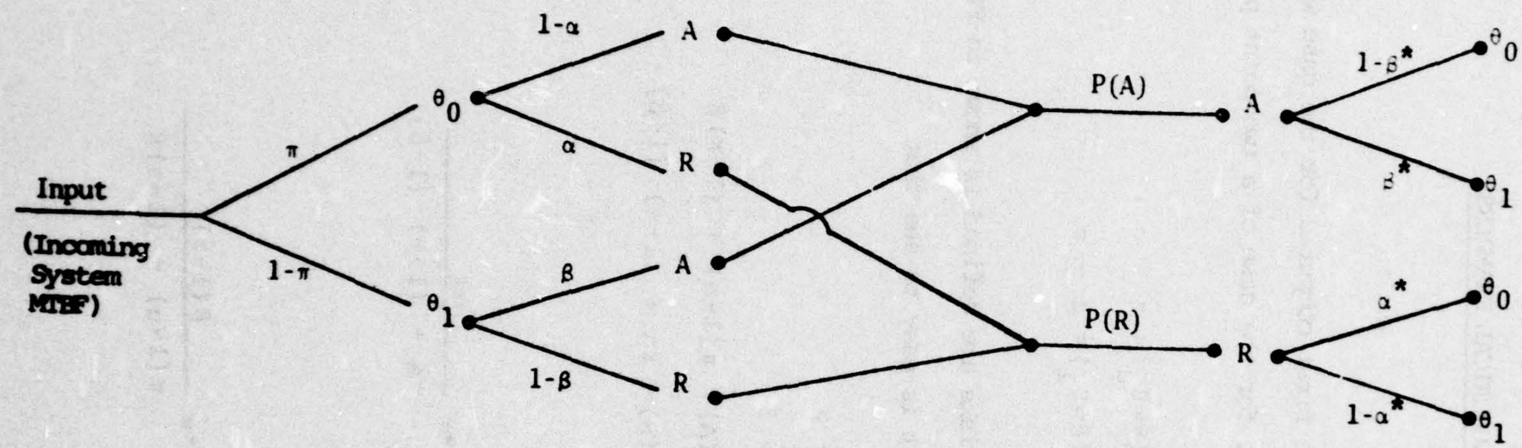

Fig. 3. SEQUENCE OF EVENTS

Let us consider two cases, one when the prior distribution of θ indicates a low probability of θ_o and the second when the probability of θ being good is high.

<u>**Case 1:**</u>

Let $\pi = 0.2$, $\alpha = \beta = 0.1$. Then,

$\bar{\alpha} = \bar{\beta} = 0.1$, $\alpha^* = 0.03$, $\beta^* = 0.307$, $P(A) = .26$, $P(R) = 0.74$

<u>**Case 2:**</u>

Let $\pi = 0.8$, $\alpha = \beta = 0.1$. Then,

$\bar{\alpha} = \bar{\beta} = 0.1$, $\alpha^* = .307$, $\beta^* = 0.03$, $P(A) = .74$, $P(R) = 0.26$

This example clearly shows that any risk may be expressed as a nonlinear function of a pair of consumer-producer risks and the prior distribution $g(\theta)$. In this example the functional relationship is explicit. In general the relationship may be nonlinear and implicit. From the results obtained above, for a good prior $\alpha^* \geq \bar{\alpha}$, $\beta^* \leq \bar{\beta}$ and for a poor prior $\alpha^* \leq \bar{\alpha}$, $\beta^* \geq \bar{\beta}$. It should be further observed that if the acceptance probability for $\theta = \theta_1$ is 0.1, it does not imply that the probability that the accepted lots have $\theta = \theta_1$ is 0.1. The questions of a proper choice of producer's and consumer's risks is, therefore, important.

Example 2:

The case of a continuous prior distribution is considered in this example. Time to failure is taken to be exponentially distributed and θ is assumed to have an inverted gamma prior of density. Equation (32). The expressions for various risks cannot be solved explicitly and, hence numerical integration is used to compute the risks. The following two cases are considered.

Case 1: $\theta_1 = 50,\ \theta_0 = 100,\ \gamma = 150,\ \lambda = 3.$

These values indicate that the prior is concentrated toward low θ values

Case 2: $\theta_1 = 50,\ \theta_0 = 100,\ \gamma = 900,\ \lambda = 7.$

These values indicate that the prior is concentrated toward high θ values.

For specified α and β, the risks $\bar{\alpha}$, $\bar{\beta}$, α^*, β^*, β^{**} and P(A) are obtained from the relationships given earlier. A listing of the various risks for various combinations of α and β is given in Table 2 for the data in Case 1 and in Table 3 for the data in Case 2. The designed values of T and r* are also given in these Tables.

TABLE 2

RELATIONSHIP BETWEEN RISKS

($\theta_1 = 50$, $\theta_0 = 100$, $\gamma = 150$, $\lambda = 3$)

α	β	$\bar{\alpha}$	$\bar{\beta}$	α^*	β^*	P(A)	β^{**}	T	r^*
0.10	0.10	0.02475	0.01728	0.00758	0.01945	0.37567	0.50348	950	13
0.10	0.15	0.02746	0.02768	0.00863	0.02990	0.39147	0.52483	790	11
0.10	0.20	0.03027	0.04626	0.00988	0.04721	0.41428	0.55229	630	9
0.10	0.25	0.02884	0.06760	0.00986	0.06494	0.44020	0.57803	540	8
0.15	0.10	0.04169	0.01529	0.01231	0.01836	0.35209	0.47941	860	11
0.15	0.15	0.03848	0.03413	0.01204	0.03712	0.38880	0.52699	660	9
0.15	0.20	0.05124	0.04796	0.01620	0.05135	0.39500	0.54059	520	7
0.15	0.25	0.04637	0.07966	0.01566	0.07768	0.43359	0.57933	422	6
0.20	0.10	0.06280	0.01622	0.01809	0.02040	0.33606	0.46660	730	9
0.20	0.15	0.06009	0.03834	0.01839	0.04322	0.37514	0.52078	540	7
0.20	0.20	0.06504	0.05151	0.02032	0.05616	0.36783	0.53891	460	6
0.20	0.25	0.09046	0.07999	0.02883	0.08459	0.39983	0.56490	318	4
0.25	0.10	0.08575	0.02150	0.02449	0.02752	0.33037	0.47071	590	7
0.25	0.15	0.09829	0.04302	0.02912	0.05132	0.35449	0.51348	420	5
0.25	0.20	0.10986	0.05961	0.03319	0.06871	0.36686	0.53592	340	4
0.25	0.25	0.12367	0.08592	0.03847	0.09432	0.3853	0.56483	260	3

TABLE 3

RELATIONSHIP BETWEEN RISKS

$(\theta_1=50,\ \theta_0=100, \gamma=900\ ,\ \lambda=7)$

α	β	$\bar{\alpha}$	$\bar{\beta}$	α^*	β^*	P(A)	β^{**}	T	r*
0.10	0.10	0.01822	0.06719	0.19790	0.00008	0.92697	0.15999	950	13
0.10	0.15	0.02065	0.09824	0.22672	0.00011	0.92774	0.16277	790	11
0.10	0.20	0.02334	0.14641	0.26181	0.00017	0.92928	0.16645	630	9
0.10	0.25	0.02250	0.19607	0.27557	0.00023	0.93521	0.17102	540	8
0.15	0.10	0.03174	0.05830	0.26122	0.00007	0.90362	0.15015	850	11
0.15	0.15	0.02987	0.11304	0.28096	0.00013	0.91565	0.15970	660	9
0.15	0.20	0.04120	0.14364	0.34076	0.00017	0.90409	0.15889	520	7
0.15	0.25	0.03777	0.21204	0.35501	0.00025	0.91559	0.16649	422	6
0.20	0.10	0.04952	0.05906	0.32584	0.00007	0.87946	0.14283	730	9
0.20	0.15	0.04855	0.11886	0.35482	0.00014	0.89146	0.15352	540	7
0.20	0.20	0.05350	0.14805	0.38302	0.00018	0.88921	0.15578	460	6
0.20	0.25	0.07792	0.19776	0.46396	0.00025	0.86678	0.15629	318	4
0.25	0.10	0.07014	0.07213	0.38941	0.00009	0.85714	0.13959	590	7
0.25	0.15	0.08331	0.12304	0.44415	0.00016	0.85122	0.14588	420	5
0.25	0.20	0.09519	0.15537	0.48034	0.00020	0.84281	0.14854	340	4
0.25	0.25	0.10978	0.19984	0.52105	0.00026	0.83288	0.15229	260	3

The following observations may be made from the results in Tables 2 and 3.

(a) In all cases, $\bar{\alpha} < \alpha$ and $\bar{\beta} < \beta$.

(b) It is almost always true that:

For Case 1 (unfavorable prior) $\alpha^* < \bar{\alpha}$ and $\beta^* > \bar{\beta}$,

For Case 2 (favorable prior) $\alpha^* > \bar{\alpha}$ and $\beta^* < \bar{\beta}$.

(c) The numerical values of the risks are considerably influenced by the parameters of the prior distributions, the maximum relative change being indicated in α^* and β^*, as should be expected.

(d) The following observations may be made regarding the nonlinear implicit relationship between the risks.

(i) For a given prior, $\bar{\alpha}$ and α^* are influenced more by changes in α than by changes in β. Similarly $\bar{\beta}$ and β^* are influenced more by changes in β than by changes in α. P(A) and β^{**} appear approximately equally affected by changes in α and β.

(ii) The relationship is nonmonotonic i.e. the same change in α and β may, depending upon the prior, lead to an increase or a decrease in other risks. This point is also clear from the values in the following table.

α	β	α^*	
		Case 1	Case 2
0.20	0.20	0.02883	0.46397
0.25	0.15	0.02912	0.44416

4. CHOICE OF RISK CRITERIA

The risk criteria defined earlier are now evaluated from the viewpoint of their meaningfulness to the consumer and the producer. The meaningfulness depends upon the kind of protection desired by the producer and the consumer in a given situation. We first define the primary interests of the producer and the consumer for a situation we have considered all along, namely, the producer produces a large number of systems which the consumer expects to buy and a prior exists, which has been agreed upon by both. We subsequently consider selection of risks in other situations that may arise in practice.

At this point it is important to distinguish between acceptance sampling and reliability demonstration. Acceptance sampling deals with a sequence of production lots or systems and interest centers on devising a satisfactory sampling plan to make accept/reject decisions regarding each lot or system. In reliability demonstration, generally, only one system or lot is available with the provision that if satisfactory reliability is demonstrated, similar future lots or systems will be manufactured and accepted subject to acceptance sampling plans. A demonstration

or qualification test is conducted to decide whether the system has the desired reliability and to make a go/no-go decision on production. An acceptance sampling test merely concerns the acceptance or rejection of a submitted system (or lot) or the production process. Consequently, a demonstration test will require more extensive testing (per system) and smaller numerical values of risks compared to acceptance sampling.

4.1 Basic Interests of the Producer and the Consumer

Let us consider the individual interests of the producer and the consumer.

The producer is primarily interested in getting the product accepted i.e. in the total probability of acceptance P(A), the corresponding risk being P(R). Once a satisfactory P(A) is obtained, the values of α, $\bar{\alpha}$, α^* may, in many cases, be of secondary importance to the producer and he would like a plan that ensures maximum probability of acceptance. However, with recently imposed comprehensive warranty agreements in many government procurements, the producer may be concerned about poor quality equipment being accepted because warranty repair and field MTBF guarantee commitments can become very costly. Furthermore, DoD as a consumer should be willing to

protect the producer who meets specifications, or to keep a low rejection rate on that part of production which meets the specifications. DoD should not be disturbed by rejection of large percentages of material below specifications. The risk $\bar{\alpha}$ concentrates on that material which meets specification and deserves protection. If the plan guarantees $\bar{\alpha}$, the producer has an incentive to keep $P(\theta \geq \theta_0)$ high in order to keep $P(A)$ high. These considerations tend to indicate that the risk $\bar{\alpha}$ could be of more interest to the producer than the risk $P(R)$.

The consumer will use the product that has been accepted and his main concern is to ensure that the accepted product contains a minimum amount of bad product, i.e. the consumer is primarily interested in the distribution of θ in the accepted product, and would like to control $f(\theta|A)$. The consumer is not satisfied with simply limiting the probability of acceptance of bad product. For example, if the entire production is bad, and if the consumer accepts 5% of the bad product under a given plan (e.g., $\bar{\beta} = 0.05$), then the consumer has 100% bad product at hand. In this sense, β and $\bar{\beta}$ do not reflect the consumer's interest adequately whereas β^* does. Ideally, the consumer would like to control the entire curve $f(\theta|A)$ (see Figure 2). However, such a specification is not practical. The consumer may then choose one of the following specifications which control one or two points on the cumulative density function of $\theta|A$.

Specification 1 (S_1) The consumer specifies a single value θ_1 and is completely satisfied if an accepted system has $\theta \geq \theta_1$. The consumer's risk is $P(\theta \leq \theta_1|A) = \beta^*$.

Specification 2 (S_2) The consumer specifies θ_1, θ_0 and the associated probabilities of $\theta \leq \theta_1$ and $\theta \leq \theta_0$ in the accepted systems. The appropriate risks are $P(\theta \leq \theta_1|A)=\beta^*$ and $P(\theta \leq \theta_0|A) = \beta^{**}$.

Specification 3 (S_3) The consumer specifies θ_1, θ_0 and the probability of $\theta \leq \theta_1$ in the accepted systems. He is, however, also interested in somehow ensuring that θ in the accepted systems is not unduly weighted towards θ_1, but is unwilling to quantify it through β^{**}. The consumer's risk is $P(\theta \leq \theta_1|A) = \beta^*$.

It should be noted that even if the consumer completely specifies $f(\theta|A)$ or $P(\theta \leq \theta'|A)$ for a given $g(\theta)$, $P(R)$ can still be independently specified. However, specification of $f(\theta|A)$, $g(\theta)$, $P(R)$ may not be compatible and/or may lead to a difficult to realize $P(\theta|A)$, resulting in the non-existence of a test plan. The approaches suggested above permit flexibility in obtaining the plan without unduly constraining the consumer's interest.

4.2 Choice of Risks for Various Practical Situations

The following cases may occur in practice.

Case 1. The producer produces a large number of systems and the consumer buys a large number of systems. This case is denoted by $P_L C_L$.

Case 2. The producer produces a large number of systems and a specific consumer buys a small number of systems ($P_L C_S$).

Case 3. A specific producer produces small number of systems and the consumer buys large number of systems (from different producers). This case is denoted by $P_S C_L$.

Case 4. Producer produces small number of systems and the consumer buys small number of systems ($P_S C_S$).

For each of these four cases, the following possibilities exist regarding $g(\theta)$.

1. A prior exists and is agreed to by the producer and the consumer.
2. The producer has a prior. The consumer does not agree with it and has no prior of his own.
3. The consumer has a prior. The producer does not agree with it and has no prior of his own.
4. Producer and consumer have different priors.

In DoD applications, Case 1 is more likely to occur. However, for the sake of completeness, we will consider all four cases.

It is suggested that every effort should be made to obtain a proper agreed upon prior either from the past data and/or from the data accumulated during testing. If a prior does not exist, there is no choice but to use the classical risks (α, β).

4.2.1 Case 1: $P_L C_L$

The producer produces large number of systems and the consumer buys large number of systems.

a. If $g(\theta)$ is sufficiently well-known and agreed to by both, then the appropriate risks for the three specifications of the consumer are:

S_1: $\{P(R), \beta^*\}$

S_2: $\{P(R), (\beta^*, \beta^{**})\}$

S_3: $\{\overline{\alpha}, \beta^*\}$

b. Sometimes the producer may have a prior distribution which assigns a large weight to values of $\theta \geq \theta_1$. The consumer may be unwilling to accept this prior but has no prior of his own. If he uses a risk β^* with this prior, he has an uncomfortably large probability of accepting systems with $\theta \leq \theta_1$, i.e. $\overline{\beta} >> \beta^*$. He does not share the producer's view regarding the small chance that such values of θ will be encountered. The consumer prefers to be protected in the classical sense by limiting $P(A|\theta = \theta_1) = \beta$ (Blumenthal, 1973).With this criterion, he may have considerable bad product at hand if most of the systems tested have a poor θ. The test should, therefore, be supplemented by requiring a review of the manufacturer's process if too many systems are rejected.

The producer believes in his prior and chooses either $\overline{\alpha}$ or $P(R)$. Thus the appropriate criteria for this situation seem to be

S_1: $\{P(R), \beta\}$

S_2: $\{P(R), \beta\}$

S_3: $\{\overline{\alpha}, \beta\}$

c. If the consumer has a prior and the producer does not, then with arguments similar to above, the appropriate criteria are

S_1: $\{\alpha, \beta^*\}$

S_2: $\{\alpha, (\beta^*, \beta^{**})\}$

S_3: $\{\alpha, \beta^*\}$

d. If both believe in different priors, the appropriate risk criteria are:

S_1: $\{P(R), \beta^*\}$

S_2: $\{P(R), (\beta^*, \beta^{**})\}$

S_3: $\{\bar{\alpha}, \beta^*\}$

with the producer's and the consumer's risks evaluated with respect to their own priors.

The suggested risk combinations for the four cases for each of the specifications S_1, S_2, and S_3 are summarized in Tables 4, 5 and 6, respectively.

4.2.2 Case 2: $P_L C_S$

The producer produces large number of systems. An individual consumer buys small (say one) number of systems.

The prior is sufficiently well-known and agreed to by both. Clearly the producer is satisfied with $\bar{\alpha}$. The consumer is primarily interested in knowing if the specific system he intends to buy is good or bad. Given a choice between $\bar{\beta}$ and β, the consumer would choose β since he would like to limit

$$\max \; P\{A|\theta \epsilon (0,\theta_1)\} = P(A|\theta_1) = \beta$$

TABLE 4

SUGGESTED RISK COMBINATIONS FOR SPECIFICATION S_1

	P_LC_L	P_LC_S	P_SC_L	P_SC_S
Prior Accepted By Both	$[P(R),\beta^*]$	$[P(R),\beta]$	$[\alpha,\beta^*]$	$[\alpha,\beta]$
Producer Has Prior, Consumer Does Not	$[P(R),\beta]$	$[P(R),\beta]$	$[\alpha,\beta]$	$[\alpha,\beta]$
Consumer has Prior Producer Does Not	$[\alpha,\beta^*]$	$[\alpha,\beta]$	$[\alpha,\beta^*]$	$[\alpha,\beta]$
Both Have Different Priors	$[P(R),\beta^*]$	$[P(R),\beta]$	$[\alpha,\beta^*]$	$[\alpha,\beta]$

TABLE 5

SUGGESTED RISK COMBINATIONS FOR SPECIFICATION S_2

	$P_L C_L$	$P_L C_S$	$P_S C_L$	$P_S C_S$
Prior Accepted By Both	$[P(R), (\beta^*, \beta^{**})]$	$[\bar{\alpha}, \beta]$	$[\alpha, (\beta^*, \beta^{**})]$	$[\alpha, \beta]$
Producer Has Prior, Consumer Does Not	$[P(R), \beta]$	$[\bar{\alpha}, \beta]$	$[\alpha, \beta]$	$[\alpha, \beta]$
Consumer Has Prior, Producer Does Not	$[\alpha, (\beta^*, \beta^{**})]$	$[\alpha, \beta]$	$[\alpha, (\beta^*, \beta^{**})]$	$[\alpha, \beta]$
Both Have Different Priors	$[P(R), (\beta^*, \beta^{**})]$	$[\bar{\alpha}, \beta]$	$[\alpha, (\beta^*, \beta^{**})]$	$[\alpha, \beta]$

TABLE 6

SUGGESTED RISK COMBINATIONS FOR SPECIFICATION S_3

	$P_L C_L$	$P_L C_S$	$P_S C_L$	$P_S C_S$
Prior Accepted By Both	$[\bar{\alpha}, \beta^*]$	$[\bar{\alpha}, \beta]$	$[\alpha, \beta^*]$	$[\alpha, \beta]$
Producer Has Prior, Consumer Does Not	$[\bar{\alpha}, \beta]$	$[\bar{\alpha}, \beta]$	$[\alpha, \beta]$	$[\alpha, \beta]$
Consumer Has Prior, Producer Does Not	$[\alpha, \beta^*]$	$[\alpha, \beta]$	$[\alpha, \beta^*]$	$[\alpha, \beta]$
Both Have Different Priors	$[\bar{\alpha}, \beta^*]$	$[\bar{\alpha}, \beta]$	$[\alpha, \beta^*]$	$[\alpha, \beta]$

Now consider the choice between β and β^*. This is rather difficult to make. The set of accepted systems may contain a single θ, say θ' and $f(\theta|A)$ is really a degenerate distribution at θ'. It, therefore, appears that the consumer would be mainly interested in sampling from $f(t|\theta')$ and protect himself by choosing the classical risk β. However, if $g(\theta)$ is well-known it seems reasonable to use $g(\theta)$ for a single decision and behave as a Bayesian, in which case β^* should be an acceptable risk. In our context, we do not consider this viewpoint and suggest the use of β.

The three other situations regarding prior can be similarly dealt with for specifications S_1, S_2, S_3, and the suggested risk combinations are given in Tables 4, 5 and 6, respectively.

4.2.3 Case 3: $P_S C_L$

The discussion is similar to Case 2 and the results are given in Tables 4, 5 and 6.

4.2.4 Case 4: $P_S C_S$

Producer produces one system and the consumer may buy it. $g(\theta)$ does not exist in the frequency sense and can only be interpreted as a degree of belief distribution. It is in this case that Bayesian and classical viewpoints differ. The former will suggest $(\bar{\alpha},\beta^*)$ or $(P(R),\beta^*)$ risks while the latter will dictate (α,β) risks. Consistent with our earlier comments, we stay with the later risks. Various risk combinations for this case are summarized in Tables 4, 5 and 6.

5. Concluding Remarks

In the preceding sections we have given the definitions, interpretations and interrelationships of the various risk combinations that arise due to the existence of a prior distribution on θ. A detailed discussion on the choice of risk criteria is presented in Section 4. In Subsection 4.2 we have suggested various risks criteria that may be appropriate for use in the design of plans under several situations. The arguments employed for suggesting these risks may not hold true in some situations and caution should be employed in using the criteria suggested in Tables 4, 5 and 6. The ultimate choice of risks to be used in a given situation will have to be determined on the basis of the particular needs of the producer and the consumer.

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